Sensitivity Analysis of Annuity Models

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Introduction

- Annuities are among the most important life insurance products.
- The cost of annuities is determined by mathematical models based on financial and demographic factors.
- For adequate risk management actions it becomes necessary an effective uncertainty quantification of factor risks: which risk has the biggest impact in determining the cost of annuities? Are there completely irrelevant factors? Do they interact?
- We present a comprehensive framework for Sensitivity Analysis (SA) of annuities based at different scales.
Annuity model

The annuity model we consider is the standard whole-life continuous annuity at age $x$

$$\bar{a}_x = \int_0^\infty t p_x \exp \left[ -\delta t \right] dt,$$

(1)

where $\delta$ represents the force of interest. The surviving probability is given by $t p_x = \exp \left[ - \int_0^t \mu_{x+s} ds \right]$, where $\mu_x$ is the force of mortality at age $x$. 
We assume that force of mortality (at time 0) at age \( x + u \) follows the Gompertz law with parameters \( b \) and \( c \)

\[
\mu^0_{x+u} = \exp[b + c(x + u)].
\] (2)

and that the mortality rates decrease by an exponential reduction function of the form \( \exp[-\alpha t] \), so that

\[
\mu^t_{x+u} = \mu^0_{x+u} e^{-\alpha t}.
\] (3)

Under these assumptions, the probability of surviving \( t \) years at age \( x \) (on a cohort basis) becomes

\[
tp_x = \exp \left[ -\mu^0_x \left( \frac{e^{(c-\alpha)t} - 1}{c - \alpha} \right) \right].
\] (4)
Consequently, the cost of annuity becomes the function

\[ \bar{a}(\mu_x^0, c, \alpha, \delta) = \int_0^\infty \exp \left[ -\mu_x^0 \left( \frac{e^{(c-\alpha)t} - 1}{c - \alpha} \right) \right] e^{-\delta t} dt \]  

(5)

with \( \alpha < c \).

A first very simple way to investigate the model is to evaluate it when inputs vary one factor at a time from a base-case input \( \mathbf{x}^0 = (\mu_x^0, c, \alpha, \delta)^0 \) to a best case \( \mathbf{x}^+ = (\mu_x^0, c, \alpha, \delta)^+ \) and to a worst case \( \mathbf{x}^- = (\mu_x^0, c, \alpha, \delta)^- \).
In the general case, denote with $g(x) : \mathbb{R}^n \to \mathbb{R}$ the input-output mapping of interest. Then, we can define the finite-change sensitivity measures [Borgonovo and Plischke, 2016]

$$\Delta^+_i y = g(x^+_i : x^0_{-i}) - g(x^0)$$

(6)

and

$$\Delta^-_i y = g(x^-_i : x^0_{-i}) - g(x^0)$$

(7)

where $(x^+_i : x^0_{-i})$ and $(x^-_i : x^0_{-i})$ denote the scenario in which the $i$–th input is changed according to the best or worst case, respectively, for $i = 1, ..., n$. 
Finite change decomposition

For any multivariate mapping it is possible to decompose the finite change $\Delta g = g(x^1) - g(x^0)$ across two different scenarios $x^0$ and $x^1$ with the finite-change ANOVA expansion [Borgonovo, 2010]

$$\Delta g = \sum_{i=1}^{n} \Delta g_i + \sum_{i<j} \Delta g_{i,j} + \sum_{i<j<k} \Delta g_{i,j,k} + \ldots + \Delta g_{1,2,\ldots,n}, \quad (8)$$

where the $2^n - 1$ finite change effects of increasing dimension are recursively given by

$$\begin{cases}
\Delta g_i = g(x^1_i : x^0_{-i}) - g(x^0) \\
\Delta g_{i,j} = g(x^1_{i,j} : x^0_{-i,j}) - \Delta g_i - \Delta g_j - g(x^0) \\
\ldots
\end{cases} \quad (9)$$

The effects $\Delta g_i, i = 1, 2, \ldots, n,$ are called main effects and the other higher order terms are the interaction effects.
Total finite-change indices

In general, given the decomposition (8), it is possible to define the total effect of factor $x_i$

$$\Delta_i^T g = \Delta g_i + \sum_{j=1, j \neq i}^{n} \Delta g_{i,j} + \sum_{k,j=1, k \neq i \neq j}^{n} \Delta g_{i,j,k} + \ldots + \Delta g_{1,2,\ldots,n}$$ (10)

which is a measure of the total impact of $x_i$ to the total change $\Delta g$. Analogously, the total interaction effect is

$$\Delta_i^I g = \Delta_i^T g - \Delta g_i$$ (11)

the difference between the total and the main effects of $x_i$. 
Local sensitivity methods provide insights only around the base point $x^0$.

Inputs have typically a range and global measures become of interest.

To explore the whole input space one can consider a series of two scenarios sampled across all the input space and then aggregate local importance measures.

Method of Elementary Effects [Morris, 1991; Campolongo et al., 2011; Borgonovo and Rabitti, ?].
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimated value</th>
<th>Lower range</th>
<th>Upper range</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_0^{60}$</td>
<td>0.00552155</td>
<td>0.005</td>
<td>0.006</td>
</tr>
<tr>
<td>$c$</td>
<td>0.085</td>
<td>0.08</td>
<td>0.09</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>-</td>
<td>-0.07</td>
<td>0.07</td>
</tr>
<tr>
<td>$\delta$</td>
<td>-</td>
<td>0%</td>
<td>10%</td>
</tr>
</tbody>
</table>

Parameter values and ranges from [Haberman et al., 2011]. They are estimated by regression using the Continuous Mortality Investigation (1991-1994) mortality table for female pensioners at ages 60 and over. [Haberman et al., 2011] introduce ranges of variation for the parameters $c$, $\alpha$ and $\delta$. We have chosen the range of $\mu_0^{60}$.
Scatterplots with $N = 10000$
Scatterplots with $N = 10000$
[Pearson, 1905] defines the correlation coefficient

\[ \eta_i^2 = \frac{\text{Cov}(Y, X_i)}{\sigma_Y \sigma_{X_i}} \]  \quad (12)

where \( \sigma_{X_i} \) is the standard deviation of the input \( X_i, \ i = 1, ..., n \). This index measures the linear dependence between the two variables \( Y \) and \( X_i \).
Hoeffding, 1948; Efron and Stein, 1981] prove that the multivariate mapping $g$ can be decomposed as

$$
g(x) = g_0 + \sum_{i=1}^{n} g_i(x_i) + \sum_{i<j} g_{i,j}(x_i, x_j) \ldots + g_{1,2,\ldots,n}(x_1, x_2, \ldots, x_n)$$

(13)

where

\[
\begin{cases}
g_0 = \int g(x) d\mu_x \\
g_i(x_i) = \int g(x) d\mu_{x\setminus i} - g_0 \\
g_{i,j}(x_i, x_j) = \int g(x) d\mu_{x\setminus i,j} - g_i(x_i) - g_j(x_j) - g_0 \\
\ldots
\end{cases}
\]

(14)
Under independence the terms $g_z(x_z)$, $z \subseteq \{1, \ldots, n\}$, are orthogonal.

The output variance $\sigma^2_Y$ can be decomposed as

$$\sigma^2_Y = \sum_{i=1}^{n} \sigma^2_i + \sum_{i<j} \sigma^2_{i,j} + \sigma^2_{1,2,\ldots,n} \quad (15)$$

where $\sigma^2_z = V[g_z(x_z)]$ is the variance of the group of variables indexed by $z \subseteq \{1, \ldots, n\}$. Every term can be interpreted as

$$\sigma^2_z = \text{Var}_{X_z} [E_{X_{-z}} [Y|X_z]] . \quad (16)$$

The index (16) has been used by [Bruno et al. 2000; Karabey et al. 2014] to study the risk of a portfolio of life insurance policies with mortality and interest rate risks.
If we normalize by the total variance, one finds

\[ \sum_{i=1}^{n} S_i + \sum_{i<j} S_{i,j} \ldots + S_{1,2,\ldots,n} = 1, \]  

(17)

where the generic term is the sensitivity index of [Sobol’, 1993] and is given by

\[ S_z = \frac{\sigma_z^2}{\sigma_Y^2}. \]  

(18)

Every term \( S_z \) measures the proportion of the output variance which the inputs \( x_z \) contribute to.
[Homma and Saltelli, 1996] define the total effect of the inputs $x_z$ as

$$S_z^T = \sum_{u \cap z \neq \emptyset} S_u.$$  \hspace{1cm} (19)

It is a measure of the total impact of inputs in $z$. The sensitivity measures $S_z$ and $S^T_z$ can shed light on the importance of the inputs $z$ in explaining the output variability.
Moment-independent sensitivity methods

[Baucells and Borgonovo, 2013] consider the sensitivity index $\beta_{i}^{KS}$

$$
\beta_{i}^{KS} = E \left[ \sup_{y} |F_{Y}(y) - F_{Y|X_{i}}(y)| \right].
$$

(20)

Suppose now that the output admits a density $f_{Y}(y)$. [Borgonovo, 2007] defines the $\delta_{i}^{BO}$ sensitivity measure

$$
\delta_{i}^{BO} = \frac{1}{2} E \left[ \int |f_{Y}(y) - f_{Y|X_{i}}(y)| dy \right].
$$

(21)

These sensitivity measures are invariant under monotonic transformations.
Figure 1: Sensitivity indices estimated from $N = 10000$ Monte Carlo runs.
[Deelstra et al., 2016; Dacorogna and Apicella, 2016] consider the role of dependence between mortality and interest rate in actuarial valuations. However, in such case there are some theoretical complications to calculate the variance-based indices [Li and Rabitz, 2017]. Nonetheless, moment-independent measures can still be computed.
Figure 2: The empirical density of the annuity model for independent (blue line), positively correlated (red line) and negatively correlated inputs (yellow line).
Figure 3: Comparison of moment-independent sensitivity measures in absence of correlation (blue bars) and with positive (yellow bars) and negative correlation (golden bars) of 0.6 between $\alpha$ and $\delta$. The Monte Carlo runs are $N = 10000$. 
Conclusions

- In the past it has been debated whether financial risk connected to life annuities is more important than the mortality risk.
- We have proposed the comprehensive framework of [Borgonovo, Plischke and Rabitti, submitted] to investigate the importance of these factors in determining the cost of annuities.
- Our results in the global case are in line with those of [Karabey et al., 2014]. Moreover, we also provide insights on the local and global scale with dependence.
- Future research: SA for stochastic simulation for portfolios of variable annuities.