Three-layer problems and the Generalized Pareto distribution

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About the speaker

- Full actuary (DAV), self-employed
- Studied Math at Univ. Munich, Pisa, Oldenburg
- Started actuarial career at Rome
- 10 years with leading reinsurers
- 10+ years as consulting actuary
- Specialized in: non-life reinsurance pricing, dealing with scarce data
Situation

Tail modelling, e.g. for layer pricing, Solvency

• Very scarce loss data

• Helpful information possibly from different sources, e.g. your portfolio vs market benchmark

• Models not fully specified

• Only easily accessible data bits:
  frequencies at thresholds / risk premiums of layers
Task: Pricing of layers from 1 up to 20 (million USD)

• A dozen large losses from your portfolio enable you to quote the layer 2 xs 1, risk premium: 1.04

For the whole market someone quoted the layer 5 xs 5, risk premium: 3

• Your portfolio supposedly has average exposure, market share is 8%, thus your risk premium for this «market» layer would be: 0.24

For higher layers you don’t have market quotations or don’t believe them

• Maximum desired payback period for large events (politically set): 200 years
General approach

• Be **modest**: no best-fit ambitions, a **good-enough** model is fine (*satisfice, don’t optimize*)

• Use Collective Model of Risk Theory

• Try to find frequency / severity that reproduce given data bits (essentially a moment matching variant)
Three-layer problems

Given input:
• Risk premiums for 3 layers
• Frequencies for 3 thresholds
• Mixed cases

Heuristics: frequency at threshold = risk rate on line of very thin layer

\[
RRoL = \frac{\text{risk premium}}{\text{limit}}
\]
MTPL Example

Formulate as (mixed) three-layer problem:

- layer 2 xs 1: $RRoL = 52\%$
- layer 5 xs 5: $RRoL = 4.8\%$
- threshold 20: $freq. = 0.5\%$
Theorem

For 3 **disjoint** layers with RRoL’s \( r_1 > r_2 > r_3 > 0 \) the problem can be solved:

by a **unique** GPD tail severity \( P(X > x|X > s) = \left( \left(1 + \xi \frac{x-s}{\sigma} \right)^+ \right)^{-\frac{1}{\xi}} \)

together with a (unique) frequency at the attachment point \( s \geq 0 \) of the lowest layer

- Works also with thresholds or mixed input
- Top layer may be unlimited
Remarks

• Easy to find numerically
• Special case: 1 layer with risk premium, layer loss frequency, and total layer loss frequency
• Single-parameter Pareto solves analogous 2-layer problems
• GPD solves many real-world 4-layer problems approximately, piecewise GPD exactly
• Results yield model-building recipes for a variety of scarce-data situations
MTPL Example

s = 1 (million USD)

• $\lambda = 1.09$
• $\xi = 0.41$  \hspace{1cm} (\alpha = 2.44)$
• $\sigma = 0.96$
Model risk

... must be high with scarce data, however:

- **Major uncertainty** is expected loss – and possibly the loss count model
- Higher moments of the severity often don’t add much further uncertainty, in particular for layers in the middle of a program
- The GPD is a choice, but a good one, both in **practical** and **statistical** sense: other severities are less handy and will often produce very similar output
Parameter-free inequality

Limited layer: limit $c$, layer loss severity $Z$, $f \geq r \geq g \geq 0$
with loss frequency $f$, total loss frequency $g$, RRoL $r$

$$1 - \frac{f - r}{f - g} \frac{r - g}{r} \leq \frac{E(Z^2)}{c \ E(Z)} \leq 1$$

• Interval is narrow for heavy tailed severity
• Narrower interval for concave cdf
The building of models by solving three-layer problems is powerful and, in case of very scarce data, an excellent trade-off between statistical ambition and the need to get things done.

Thanks for joining this talk.
Feedback welcome, now or later.

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