Fat-tailed Distributions for Investment Variables

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Use “Wilkie model”

Fit model for each variable assuming normal residuals.

Take these residuals and fit different distributions.
Introduction

➢ Possible distributions

1. Normal
2. Laplace (double exponential)
3. Skew Laplace
4. Hyperbolic
5. Skew hyperbolic

➢ All examples of “conical distributions”
Conical Distributions

- Take some conic section: parabola, hyperbola, two straight lines
- Limit to those with full range for $x$, $-\infty$ to $+\infty$ 
  omit circle, ellipse, etc
- Arrange as $y = g(x)$
- Choose that part with $y < 0$
- Put $h(x) = \exp(y)$
- Take density $f(x) = k.h(x)$
- Find $k$ so that $\int f(x).dx = 1$
  i.e. find $1 / k = \int h(x).dx$
Conical Distributions

- Parabola with nose at (0, 0) axis vertical

- This gives **Normal distribution**

\[ y = -ax^2 \]
\[ f(x) = k \cdot \exp(-ax^2) \]
\[ \mu = 0 \quad 1/a = 2\sigma^2 \quad 1/k = \sigma \sqrt{2\pi} \]
Conical Distributions

- Two straight lines, symmetric, crossing at (0, 0)
  - Laplace, two symmetric exponentials
    \[ f(x) = \alpha \cdot \exp(-\text{abs}(\alpha x)) / 2 \]
    \[ k = \alpha / 2 \]
    \[ 0 < \alpha \]

Often parameterised with \( \lambda = 1/\alpha \)
Conical Distributions

- Two straight lines, skewed, crossing at (0, 0)
- **Skew Laplace**, two different exponentials, meeting at $x = 0$

$$f(x) = k \cdot \exp(\alpha(1+\rho)x) \quad x < 0$$

$$= k \cdot \exp(-\alpha(1-\rho)x) \quad x > 0$$

$$k = \frac{\alpha(1 - \rho^2)}{2}$$

$$0 < \alpha \quad -1 < \rho < +1$$

- Could be parameterised with $\lambda_1, \lambda_2$
Conical Distributions

- Hyperbola with main axis vertical
  asymptotes crossing at (0, 0) symmetric

- Gives hyperbolic
  \[ f(x) = k \cdot \exp(-\alpha \delta \sqrt{1 + (x/\delta)^2}) \]
  \[ 1/k = 2\delta.K_1(\alpha \delta) \]
  \[ K_1(.) \text{ is one of the Bessel functions} \]
  \[ 0 < \delta \quad 0 < \alpha \]
Conical Distributions

- Skew hyperbola gives skew hyperbolic
  \[ f(x) = k \exp (-\alpha \delta \sqrt{1 + (x/\delta)^2} + \rho x/\delta) \]

- Put \( \gamma = \alpha \sqrt{1 - \rho^2} \)
  \[ 1/k = 2\alpha \delta K_1(\gamma \delta) / \gamma \]
  \( K_1(.) \) as before a Bessel function

- Often parameterised differently

- \( 0 < \delta \quad 0 < \alpha \quad -1 < \rho < +1 \)
Conical Distributions

- All can be offset from (0, 0) to (μ, 0)

- For symmetric versions this gives mean μ

- For skew versions, mean depends on parameters

- There is a scale factor for each, e.g. σ, 1/α, δ
Conical Distributions

Conical functions

-6 -5.5 -5 -4.5 -4 -3.5 -3 -2.5 -2 -1.5 -1 -0.5 0 0.5 1 1.5 2 2.5 3 3.5 4 4.5 5 5.5 6

-8 -7 -6 -5 -4 -3 -2 -1 0

- Parabola
- Two straight lines
- Two skew lines
- Hyperbola
- Skew hyperbola
Conical Distributions

Density functions, all with mean 0, variance 1

- Normal
- Laplace
- Skew Laplace
- Hyperbolic
- Skew Hyperbolic
Conical Distributions

Distribution functions

- **Normal**
- **Laplace**
- **Skew Laplace**
- **Hyperbolic**
- **Skew Hyperbolic**
Conical Distributions

- Limiting versions:
  - If \( \rho = -1 \) or \( \rho = +1 \)
  - One of the straight lines (or asymptotes) becomes the vertical axis.
  - No longer full range of \( x \)

- For hyperbola:
  - if \( \alpha = 0 \) we get two straight lines
  - if \( \alpha = \infty \) we get parabola
The Wilkie Model – Principal Variables

- Consumer prices index, $Q$
- Wages index, $W$
- Share dividends, $D$
- Share dividend yield, $Y = D/P$
- Share earnings, $E$
- Cover, $V = E/D$
- Multiple, $M = P/E$ ratio
- Long interest rate, $C$
- Short term interest rate, $B$
- Real yield on index-linked bonds, $R$
The Wilkie Model - Residuals

- For each series $x$, $xZ$ is the standardised residual, i.e.
  \[
  \text{Normal (0, 1)}
  \]

- Consider first Skewness and Kurtosis
<table>
<thead>
<tr>
<th>Series</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>p(J-B)</th>
<th>Normal?</th>
</tr>
</thead>
<tbody>
<tr>
<td>QZ</td>
<td>1.31</td>
<td>6.39</td>
<td>0.0</td>
<td>No</td>
</tr>
<tr>
<td>WZ</td>
<td>0.39</td>
<td>3.92</td>
<td>0.0526</td>
<td>Possibly</td>
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<tr>
<td>YZ</td>
<td>0.35</td>
<td>3.57</td>
<td>0.1998</td>
<td>Yes</td>
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<tr>
<td>DZ</td>
<td>-0.74</td>
<td>4.22</td>
<td>0.0006</td>
<td>No</td>
</tr>
<tr>
<td>EZ</td>
<td>-0.29</td>
<td>10.30</td>
<td>0.0</td>
<td>No</td>
</tr>
<tr>
<td>VZ</td>
<td>0.37</td>
<td>3.67</td>
<td>0.3152</td>
<td>Yes</td>
</tr>
<tr>
<td>MZ</td>
<td>-0.64</td>
<td>4.57</td>
<td>0.0081</td>
<td>No</td>
</tr>
<tr>
<td>CZ</td>
<td>-0.75</td>
<td>5.50</td>
<td>0.0</td>
<td>No</td>
</tr>
<tr>
<td>BZ</td>
<td>-3.61</td>
<td>25.98</td>
<td>0.0</td>
<td>No</td>
</tr>
<tr>
<td>RZ</td>
<td>-0.34</td>
<td>2.43</td>
<td>0.5422</td>
<td>Yes</td>
</tr>
</tbody>
</table>
Compare log-likelihood for other distribution with log-likelihood for Normal
## Comparison of log-likelihoods for different dist.

<table>
<thead>
<tr>
<th>Series</th>
<th>L–N</th>
<th>SL–N</th>
<th>H–N</th>
<th>SH–N</th>
</tr>
</thead>
<tbody>
<tr>
<td>QZ</td>
<td>10.12</td>
<td>10.65</td>
<td>10.15</td>
<td>10.71</td>
</tr>
<tr>
<td>WZ</td>
<td>1.17</td>
<td>1.23</td>
<td>1.96</td>
<td>2.97</td>
</tr>
<tr>
<td>YZ</td>
<td>−3.23</td>
<td>−1.12</td>
<td>0.30</td>
<td>0.84</td>
</tr>
<tr>
<td>DZ</td>
<td>2.28</td>
<td>4.98</td>
<td>2.69</td>
<td>4.98</td>
</tr>
<tr>
<td>EZ</td>
<td>8.51</td>
<td>8.65</td>
<td>8.73</td>
<td>8.78</td>
</tr>
<tr>
<td>VZ</td>
<td>0.25</td>
<td>0.54</td>
<td>0.95</td>
<td>1.34</td>
</tr>
<tr>
<td>MZ</td>
<td>0.03</td>
<td>2.02</td>
<td>1.31</td>
<td>2.04</td>
</tr>
<tr>
<td>CZ</td>
<td>4.88</td>
<td>6.31</td>
<td>5.06</td>
<td>6.48</td>
</tr>
<tr>
<td>BZ</td>
<td>25.85</td>
<td>26.22</td>
<td>25.87</td>
<td>26.22</td>
</tr>
<tr>
<td>RZ</td>
<td>−1.69</td>
<td>−1.25</td>
<td>−0.01</td>
<td>0.65</td>
</tr>
</tbody>
</table>
Skewness-Kurtosis Diagram

- Skewness-Kurtosis (S - K) diagram
- Normal (S, K) = (0, 3)
- Laplace (S, K) = (0, 6)
- Skew Laplace varies with \( \rho \) on a line from \((-2, 9)\) to \((0, 6)\) to \((+2, 9)\)
- Hyperbolic, \( S = 0 \), \( K \) varies with \( \alpha \) on a line \((0, 3)\) to \((0, 6)\)
- Skew Hyperbolic varies with \( \alpha \) and \( \rho \) within a ‘triangular’ area
Skewness-Kurtosis (S-K) Diagram
S-K Diagram with points for Actual Values
S-K Diagram - points for Actual and Fitted Values

- Skew Laplace
- Hyperbolic
- Skew Hyperbolic Rho = 0.25
- Skew Hyperbolic Rho = 0.99

- Actual data
- Fitted distributions
Comparison of log-likelihoods for different dist.

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<td>WZ</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>YZ</td>
<td>–3.23</td>
<td>–1.12</td>
<td>0.30</td>
<td>0.84</td>
</tr>
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<td>DZ</td>
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<td></td>
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</tr>
<tr>
<td>RZ</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Conclusions

- Normal: $YZ, VZ, RZ$
- Laplace: $BZ, EZ$
- Skew Laplace: $DZ$
- Hyperbolic: none
- Skew Hyperbolic: $QZ, WZ, CZ, MZ$
The Wilkie Model – Retail Prices

Actual and fitted densities, QZ

- Actual
- Normal
- Laplace
- Skew Laplace
- Hyperbolic
- Skew Hyperbolic

Density

X Value
The Wilkie Model – Simulations

- Simulations
  Normal as usual (Marsaglia’s method)
  Laplace by inversion
  Hyperbolic acceptance/rejection method

- Principal variables:
  \( Q, W, D, P, E \) index-type
  \( Y, V, M, C, B, R \) ratio-type

- Total Return indices, including income
  \( PT, CT, BT, RT \) index-type
The Wilkie Model – Simulations

Compound continuous rate of total return

\[ GQL(t) = \frac{\ln(Q(t)) - \ln(Q(0))}{t} \]

\[ = \frac{QL(t) - QL(0)}{t} \]

Nominal

\[ GWL(t) = \frac{WL(t) - WL(0)}{t} \]

‘Real’ rate

\[ GWLR(t) = GWL(t) - GQL(t) \]
The Wilkie Model – Simulations

- 1,000,000 simulations for 50 years.
- Very large amounts of output.

- Criteria based on quantiles $Q(a)$
  \[
  \frac{1}{2} (Q(1 - a) - Q(a)) / \text{Standard Deviation}
  \]

  - $a = 5\%$ gives 90% spread, C 90%
  - $a = 0.5\%$ gives 99% spread, C 99%

Normal C 90% = 1.64 and C 99% = 2.58
The Wilkie Model – Simulations

Retail Prices, $GQL(t)$

Results with Skew Hyperbolic innovations

<table>
<thead>
<tr>
<th>Term, $t$</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>Skewness</td>
<td>0.59</td>
<td>0.44</td>
<td>0.28</td>
<td>0.19</td>
<td>0.13</td>
<td>0.08</td>
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<tr>
<td>Kurtosis</td>
<td>5.99</td>
<td>4.76</td>
<td>3.69</td>
<td>3.32</td>
<td>3.16</td>
<td>3.06</td>
</tr>
<tr>
<td>C 90%</td>
<td>1.62</td>
<td>1.62</td>
<td>1.63</td>
<td>1.64</td>
<td>1.64</td>
<td>1.64</td>
</tr>
<tr>
<td>C 99%</td>
<td>3.19</td>
<td>2.99</td>
<td>2.77</td>
<td>2.67</td>
<td>2.62</td>
<td>2.60</td>
</tr>
</tbody>
</table>

- Kurtosis reduces with $t$ so does Skewness
- C 90% same as Normal
- C 99% larger than Normal, reduces with $t$
Most variables like Normal with varying Kurtosis in year 1

But Long-term interest rates, $C$, with $GCTR(t)$ and Short-term interest rates, $B$, with $GBTR(t)$ different, because basic model mixes Normal and Lognormal so even with Normal innovations there is very high Kurtosis.
Future Work

- Still to do:
  - Re-estimate all the parameters of all the variables with an appropriate new distribution
  - For Retail Prices, Skew Laplace becomes best
REFERENCES


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- Wilkie, A.D. & Şahin Ş. (in draft). Yet more on a stochastic economic model: Part 6B, Investigating distributions for residuals using the Normal parameters for the skeleton