The Application of Affine Processes in Cohort Mortality Risk Models
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Introduction

- Mortality models have attracted research attention over recent years because of the significance of longevity risks (The Joint Forum (2013)) and capital requirements for insurers (Barrieu et al. (2012)).

- Continuous time mortality models (Milevsky and Promislow (2001), Dahl (2004), Cairns et al. (2006a), Blackburn and Sherris (2013)) have received less attention than discrete time mortality models (Lee and Carter (1992), Cairns et al. (2009), Cairns et al. (2006b), Renshaw and Haberman (2006)).

- Continuous time affine cohort mortality models have attracted more recent research (Dahl and Møller (2006), Biffis (2005), Luciano et al. (2008), Schrager (2006), Jevtic et al. (2013), Xu et al. (2015), Chang and Sherris (2018)).
Research motivation

- We propose an affine mortality model based on the Arbitrage-Free Nelson-Siegel (AFNS) yield curve model (Christensen et al. (2011)) with identifiable factors of level, slope and curvature of the mortality curve.

- We assess a number of continuous-time affine cohort mortality models (Blackburn-Sherris dependent and independent factor mortality models, AFNS independent and dependent factor mortality models, CIR mortality model).

- We investigate the impact of incorporating factor dependence to capture age correlations for the models.

We capture cohort effects directly using age-cohort data to calibrate and assess the model survival curves fit and forecasting performance.
The dynamics of the latent factors $X_t$ are given by the following system of stochastic differential equations (SDEs) under the risk-neutral measure $Q$ (Duffie and Kan, 1996; Christensen et al., 2011):

$$dX_t = K^Q \left[ \theta^Q - X_t \right] dt + \Sigma D(X_t, t) dW^Q_t,$$

(1)

where $K^Q \in \mathbb{R}^{n \times n}$ is the mean reversion matrix, $\theta^Q \in \mathbb{R}^n$ is the long-term mean, $\Sigma \in \mathbb{R}^{n \times n}$ is the volatility matrix, $W^Q_t \in \mathbb{R}^n$ is a standard Brownian motion, and $D(X_t, t)$ is a diagonal matrix with the $i$th diagonal entry as $\sqrt{\alpha^i(t) + \beta_1^i(t) x^1_t + \ldots + \beta_n^i(t) x^n_t}$. $\alpha$ and $\beta$ are bounded continuous functions.
Affine Mortality Models

- Under these dynamics the risk-neutral survival probabilities for any age $x$ from time $t$ to time $T$ can be represented as (Blackburn and Sherris, 2013):

\[
S(t, T) = \exp \left( B(t, T)' X_t + A(t, T) \right),
\]

where $B(t, T)$ and $A(t, T)$ are the solutions to the following system of ordinary differential equations (ODEs):

\[
\frac{dB(t, T)}{dt} = \rho_1 + (K^Q)' B(t, T),
\]

\[
\frac{dA(t, T)}{dt} = -B(t, T)' K^Q \theta^Q - \frac{1}{2} \sum_{j=1}^{3} \left( \Sigma' B(t, T) B(t, T)' \Sigma \right)_{j,j},
\]

with boundary conditions $B(T, T) = A(T, T) = 0$. 

Multi-Factor Affine Cohort Mortality Models

- The form of risk premium is (Duffee, 2002):

\[ \Lambda_t = \begin{cases} 
\lambda^0 + \lambda^1 X_t, & \text{models with Gaussian processes;} \\
D(X_t, t) \lambda^0, & \text{the CIR model,}
\end{cases} \tag{5} \]

where \( \Lambda_t \in \mathbb{R}^{n \times 1} \), \( \lambda^0 \in \mathbb{R}^{n \times 1} \) and \( \lambda^1 \in \mathbb{R}^{n \times n} \).

- With these assumptions, the SDEs for factors under the measure \( P \) can be written as:

\[ dX_t = \begin{cases} 
K^P \left[ \theta^P - X_t \right] dt + \Sigma dW^P_t, & \text{models with Gaussian processes;} \\
K^P \left[ \theta^P - X_t \right] dt + \Sigma D(X_t, t) dW^P_t, & \text{the CIR model,}
\end{cases} \tag{6} \]
Multi-Factor Affine Cohort Mortality Models

The dynamics of the factors in each model that we will estimate from historical mortality data is:

- The independent Blackburn-Sherris model (Blackburn and Sherris, 2013)

\[
\begin{pmatrix}
    \frac{dX_1^1}{dt} \\
    \frac{dX_2^2}{dt} \\
    \frac{dX_3^3}{dt}
\end{pmatrix} = - \begin{pmatrix}
    \delta_{11} & 0 & 0 \\
    0 & \delta_{22} & 0 \\
    0 & 0 & \delta_{33}
\end{pmatrix} \begin{pmatrix}
    X_1^1 \\
    X_2^2 \\
    X_3^3
\end{pmatrix} dt + \begin{pmatrix}
    \sigma_{11} & 0 & 0 \\
    0 & \sigma_{22} & 0 \\
    0 & 0 & \sigma_{33}
\end{pmatrix} \begin{pmatrix}
    dW_{1,Q}^1 \\
    dW_{2,Q}^2 \\
    dW_{3,Q}^3
\end{pmatrix}.
\]

(7)

- The independent AFNS model (Christensen et al., 2011) The dynamics of the factors under the $Q$-measure are given by:

\[
\begin{pmatrix}
    \frac{dL_t}{dt} \\
    \frac{dS_t}{dt} \\
    \frac{dC_t}{dt}
\end{pmatrix} = - \begin{pmatrix}
    0 & 0 & 0 \\
    0 & \delta & -\delta \\
    0 & 0 & \delta
\end{pmatrix} \begin{pmatrix}
    L_t \\
    S_t \\
    C_t
\end{pmatrix} dt + \begin{pmatrix}
    \sigma_{11} & 0 & 0 \\
    0 & \sigma_{22} & 0 \\
    0 & 0 & \sigma_{33}
\end{pmatrix} \begin{pmatrix}
    dW_{1,Q}^1 \\
    dW_{2,Q}^2 \\
    dW_{3,Q}^3
\end{pmatrix}.
\]

(8)

- The dependent Blackburn-Sherris model

\[
\begin{pmatrix}
    \frac{dX_1^1}{dt} \\
    \frac{dX_2^2}{dt} \\
    \frac{dX_3^3}{dt}
\end{pmatrix} = - \begin{pmatrix}
    k_{11}^P & 0 & 0 \\
    0 & k_{22}^P & 0 \\
    0 & 0 & k_{33}^P
\end{pmatrix} \begin{pmatrix}
    X_1^1 \\
    X_2^2 \\
    X_3^3
\end{pmatrix} dt + \begin{pmatrix}
    \sigma_{11} & 0 & 0 \\
    \sigma_{21} & \sigma_{22} & 0 \\
    \sigma_{31} & \sigma_{32} & \sigma_{33}
\end{pmatrix} \begin{pmatrix}
    dW_{1,P}^1 \\
    dW_{2,P}^2 \\
    dW_{3,P}^3
\end{pmatrix}.
\]

(9)
Multi-Factor Affine Cohort Mortality Models

- The dependent AFNS model

\[
\begin{pmatrix}
    \frac{dX_1^1}{dt} \\
    \frac{dX_2^2}{dt} \\
    \frac{dX_3^3}{dt}
\end{pmatrix}
= - \begin{pmatrix}
    \delta_{11} & 0 & 0 \\
    \delta_{21} & \delta_{22} & 0 \\
    \delta_{31} & \delta_{32} & \delta_{33}
\end{pmatrix}
\begin{pmatrix}
    X_1^1 \\
    X_2^2 \\
    X_3^3
\end{pmatrix}
\ dt + \begin{pmatrix}
    \sigma_{11} & 0 & 0 \\
    \sigma_{21} & \sigma_{22} & 0 \\
    \sigma_{31} & \sigma_{32} & \sigma_{33}
\end{pmatrix}
\begin{pmatrix}
    dW_{1^1,Q}^1 \\
    dW_{2^2,Q}^2 \\
    dW_{3^3,Q}^3
\end{pmatrix}.
\]  

(10)

- The CIR model

\[
\begin{pmatrix}
    \frac{dX_1^1}{dt} \\
    \frac{dX_2^2}{dt} \\
    \frac{dX_3^3}{dt}
\end{pmatrix}
= - \begin{pmatrix}
    \delta_{11} & 0 & 0 \\
    0 & \delta_{22} & 0 \\
    0 & 0 & \delta_{33}
\end{pmatrix}
\begin{pmatrix}
    \theta_1^Q \\
    \theta_2^Q \\
    \theta_3^Q
\end{pmatrix}
- \begin{pmatrix}
    X_1^1 \\
    X_2^2 \\
    X_3^3
\end{pmatrix}
\ dt
+ \begin{pmatrix}
    \sigma_{11} & 0 & 0 \\
    0 & \sigma_{22} & 0 \\
    0 & 0 & \sigma_{33}
\end{pmatrix}
\begin{pmatrix}
    \sqrt{X_1^1} \\
    \sqrt{X_2^2} \\
    \sqrt{X_3^3}
\end{pmatrix}
\begin{pmatrix}
    dW_{1^1,Q}^1 \\
    dW_{2^2,Q}^2 \\
    dW_{3^3,Q}^3
\end{pmatrix}.
\]  

(11)
Multi-Factor Affine Cohort Mortality Models

\( \tau = T - t \) times \( \bar{\mu}^i (x; t, T) \) is given by \(-[B(t, T)' X_t + A(t, T)]\) where \( B(t, T) \), the factor loadings, and \( A(t, T) \) have explicit expressions.

- The independent Blackburn-Sherris model (Blackburn and Sherris, 2013)

\[
B^j(t, T) = \frac{1 - e^{-\delta_{jj}(T-t)}}{\delta_{jj}}, \quad j = 1, 2, 3, \tag{12}
\]

\[
A(t, T) = \frac{1}{2} \sum_{j=1}^{3} \frac{\sigma_{jj}^2}{\delta_{jj}^3} \left[ \frac{1}{2} \left( 1 - e^{-2\delta_{jj}(T-t)} \right) - 2 \left( 1 - e^{-\delta_{jj}(T-t)} \right) + \delta_{jj}(T-t) \right]. \tag{13}
\]
Multi-Factor Affine Cohort Mortality Models

- The independent AFNS model (Christensen et al., 2011)

\[ B^1(t, T) = -(T - t), \quad B^2(t, T) = -\frac{1 - e^{-\delta(T-t)}}{\delta}, \]
\[ B^3(t, T) = (T - t) e^{-\delta(T-t)} - \frac{1 - e^{-\delta(T-t)}}{\delta}, \]

\[ \frac{A(t, T)}{T-t} = \sigma^2_{11} \frac{(T - t)}{6} + \sigma^2_{22} \left[ \frac{1}{2\delta^2} - \frac{1}{\delta^3} \frac{1 - e^{-\delta(T-t)}}{T - t} \right] + \sigma^2_{33} \left[ \frac{1}{2\delta^2} + \frac{1}{\delta^2} e^{-\delta(T-t)} - \frac{1}{4\delta} (T - t) e^{-2\delta(T-t)} - \frac{3}{4\delta^3} e^{-2\delta(T-t)} \right] - \frac{2}{\delta^3} \frac{1 - e^{-\delta(T-t)}}{T - t} + \frac{5}{8\delta^3} \frac{1 - e^{-2\delta(T-t)}}{T - t} \].
Mortality Data

- US mortality age-cohort data from the Human Mortality Database (2017) (HMD) to calibrate and compare the mortality models.
- Mortality data of males from ages 50 to 100 for the cohorts born from 1883 to 1915.
- Historical survival probability, \( S^i(x; t, T) \), and the historical average forces of mortality \( \bar{\mu}^i(x; t, T) \) over the period \( \tau = T - t \) for each cohort \( i \) aged \( x \) at time \( t \) from the data, using:

\[
S^i(x; t, T) = \prod_{s=1}^{T-t} \left[ 1 - q^i(x + s - 1, t + s - 1) \right],
\]

(16)

\[
\bar{\mu}^i(x; t, T) = -\frac{1}{T - t} \log \left[ S^i(x; t, T) \right],
\]

(17)

where \( q^i(x, t) \) is the one year death probability for an individual aged \( x \) at time \( t \) in cohort \( i \).
Figure 1: Average Force of Mortality for Males Born from 1883 to 1915

Figure 2: Fractions (%) of Variance Explained by Each of the First 7 Principal Components
## Table 1: Comparison of Affine Mortality Models

<table>
<thead>
<tr>
<th></th>
<th>The Blackburn-Sherris Model</th>
<th>The AFNS Model</th>
<th>The CIR Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Independent-</td>
<td>Dependent-</td>
<td>Independent-</td>
</tr>
<tr>
<td></td>
<td>Factor</td>
<td>Factor</td>
<td>Factor</td>
</tr>
<tr>
<td>Log Likelihood</td>
<td>9896.419</td>
<td>9938.696</td>
<td>9665.801</td>
</tr>
<tr>
<td>No. of Parameters</td>
<td>12</td>
<td>18</td>
<td>10</td>
</tr>
<tr>
<td>AIC</td>
<td>-19570.837</td>
<td>-19643.392</td>
<td>-19113.602</td>
</tr>
<tr>
<td>BIC</td>
<td>-18968.292</td>
<td>-19008.277</td>
<td>-18521.914</td>
</tr>
<tr>
<td>Probability of Negative Mortality</td>
<td>0.02700</td>
<td>1.011e-32</td>
<td>1.722e-31</td>
</tr>
</tbody>
</table>
Model Residual Analysis

(a) The Independent Blackburn-Sherris Model

(b) The Dependent Blackburn-Sherris Model

(c) The Independent AFNS Model

(d) The Dependent AFNS Model

(e) The CIR Model
MAPE of Affine Mortality Models

Figure 4: The Models with Gaussian Processes
MAPE of Affine Mortality Models

Figure 5: The CIR Model, the Dependent Blackburn-Sherris Model and the Independent AFNS Model
Table 2: RMSE by Comparing the Actual and Best-Estimate Survival Probabilities of the 1916 Cohort

<table>
<thead>
<tr>
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<tbody>
<tr>
<td></td>
<td>Independent</td>
<td>Dependent</td>
<td>Independent</td>
</tr>
<tr>
<td>RMSE</td>
<td>0.03197</td>
<td>0.00726</td>
<td>0.00668</td>
</tr>
</tbody>
</table>
Forecast RMSE

Figure 6: Actual and Best-Estimate Survival Probabilities of the 1916 Cohort

Figure 7: Absolute Percentage Errors between Actual and Best-Estimate Survival Probabilities
Summary and Conclusions

- We introduce an AFNS mortality model with interpretable latent stochastic factors for level, slope and curvature of the survival curve.


- The CIR mortality model has the best in-sample model fit reflecting the more realistic assumption of Gamma-distributed mortality rates.

- Independent-factor AFNS mortality model is parsimonious, can better capture the variation in cohort mortality rates in US data, a better fit at older ages than the independent-factor Blackburn-Sherris model, better predictive performance. Negative mortality rates have very low probability. Intuitive factor interpretation and well suited for financial and insurance applications.
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